Interferometry with Two Telescopes: Mathematical Mechanics

Bob Thompson

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Overview

- Stellar angular sizes
- The right kind of telescope
- The Michelson Stellar Interferometer
- Coherence and Optical Delay
- Combination of EM waves
- Fringes and visibility
- The visibility function and model geometries
- Calibrators

Stellar Angular Sizes
(Back of the envelope)

- Use the sun as our prototype
- Solar vs. bright star apparent brightness:
- ΔV = -26 : $V_{\Theta} - V_{*} = -2.5\log\left(I_{\Theta}/I_{*}\right)$

$\rightarrow$ 2.5 × 10^{10} change in apparent brightness

- Since brightness scales with disk area:

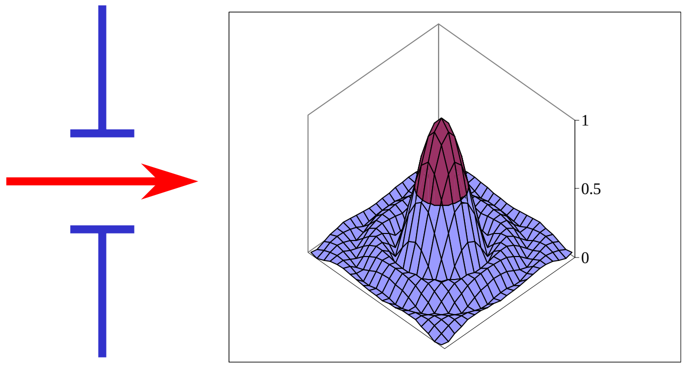
$$\frac{I_{\Theta}}{I_{*}} = \frac{A_{\Theta}}{A_{*}} = \frac{\omega_{\Theta}}{\omega_{*}} = \left(\frac{\theta_{\Theta}}{\theta_{*}}\right)^2 \rightarrow \theta_{*} = \theta_{\Theta} \times \sqrt{I_{*}/I_{\Theta}}$$

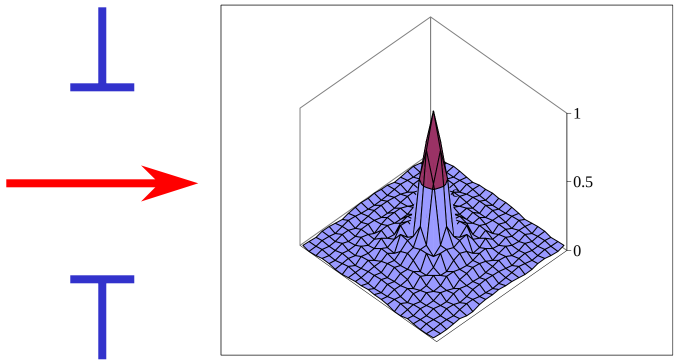
Since the sun is ~30’ = 1800” $\rightarrow$ θ_{*} = 12 mas

- Newton realized stars were of this size order (mas)

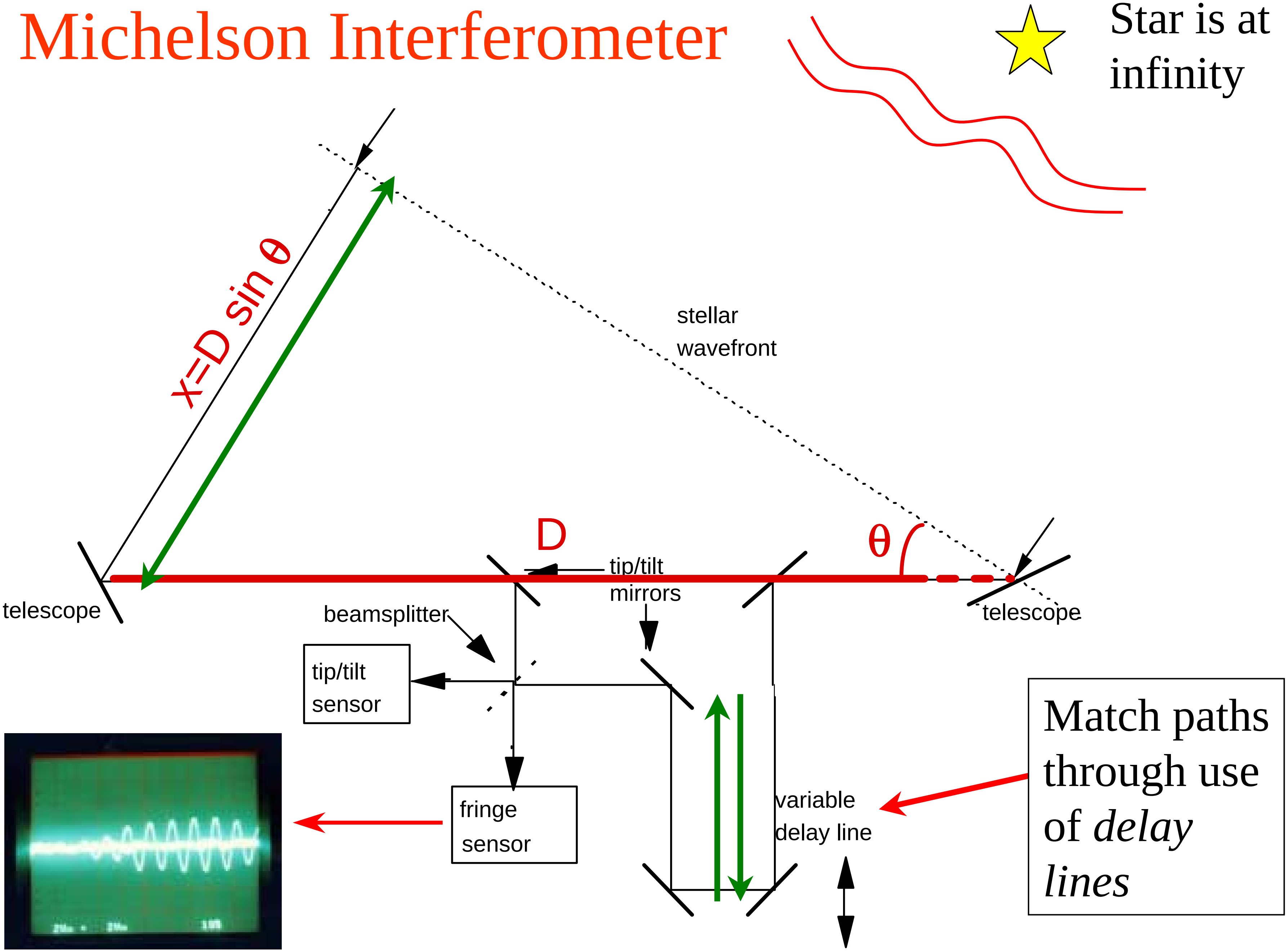
Right Kind of Telescope

- Telescope resolution goes as λ / D *(radians)*
- Very largest stars are ~50 mas
  - Can get away with D~2m in the visible (8m in K-band)
- Typical parameters are λ~1μm, θ~1 mas
  - Need D~100m
  - Leads us to interferometry





Michelson Interferometer



Coherence of the fringes

The coherence time for a fringe is the length of time before a significant phase glitch occurs.

$$\tau_0 = \frac{1}{\Delta \nu}$$

Likewise, this can be viewed in terms of a coherence length.

$$l_c = c\tau_0$$
$$= \frac{c}{\Delta \nu} = \left| \frac{\lambda^2}{\Delta \lambda} \right|$$

Using a Johnson K-band filter (λ=2.2 μm, $\Delta\lambda$=0.4 μm), the coherence length for the fringe is 12 μm. ***The smaller the bandwidth, the larger the coherence time.***

The Optical Delay

Since the Earth turns about its axis, the stars in the sky move with respect to the interferometer. This motion must be compensated by the optical delay line, such that

$$\frac{d}{dt}(OD) = B\cos(z)\frac{dz}{dt}\cos L$$

With *OD* = optical delay, *B* = projected baseline , *z* = the zenith angle of the star and *L* is the latitude of the interferometer.

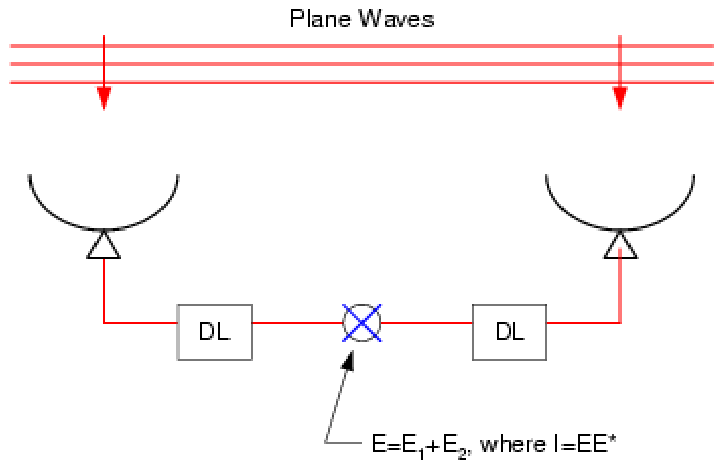
Near zenith (z = 0 deg), the maximum motion (at the equator) is about dz/dt = 7.3 x 10^{-5} rad/sec.

For a baseline of 38 m at latitude 31^{o}, the change in OD is

2.4 mm/sec.

This rate of change must be compensated to an accuracy of the coherence length, l_c.

Combination of two EM waves



Consider two EM waves:

$$E_1 = a_1 \exp(ik{*}OP_1 - i\omega t) \qquad\qquad E_2 = a_2 \exp(ik{*}OP_2 - i\omega t)$$

Combining the two waves physically means taking the complex conjugate mathematically:

$$I = E_1 E_2^* = I_T = I_1 + I_2 + 2\sqrt{I_1 I_2}\ \cos(k\Delta)$$

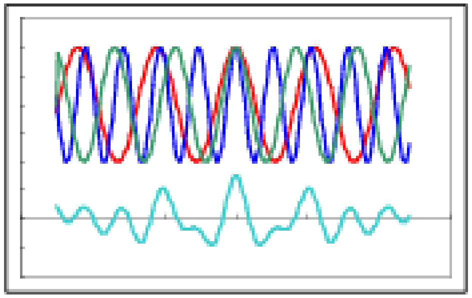
$$\text{where } \Delta = (OP_2 - OP_1)$$

Monochromatic Point Response Function

Now, assume the intensities from both apertures are equal ($I_1 = I_2 = I$). Then, the point-response function for a source at infinity becomes

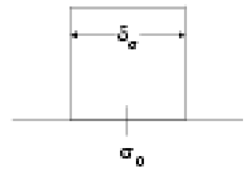
$$\mathbf{R_\lambda^P (\Delta) \equiv \frac{I_T}{2\,I} - 1 = \cos (k\Delta).}$$

However, this is a perfect-world scenario. The light collected by a real interferometer is not monochromatic (infinitely small bandwidth), but polychromatic.



Effects of Spectral Bandwidth

Consider a rectangular filter of width δ_σ and center σ_0:



Average the response Over the bandwidth:

$$\langle R^P \rangle = \frac{\int R_\sigma F d\sigma}{\int d\sigma}$$

Use simple filter function *F:*

$$= \frac{1}{\delta_\sigma} \int_{\sigma_0 - \delta_\sigma/2}^{\sigma_0 + \delta_\sigma/2} R_\sigma d\sigma$$

$$= \frac{1}{2\pi\delta_\sigma} \sin(2\pi\sigma\Delta)$$

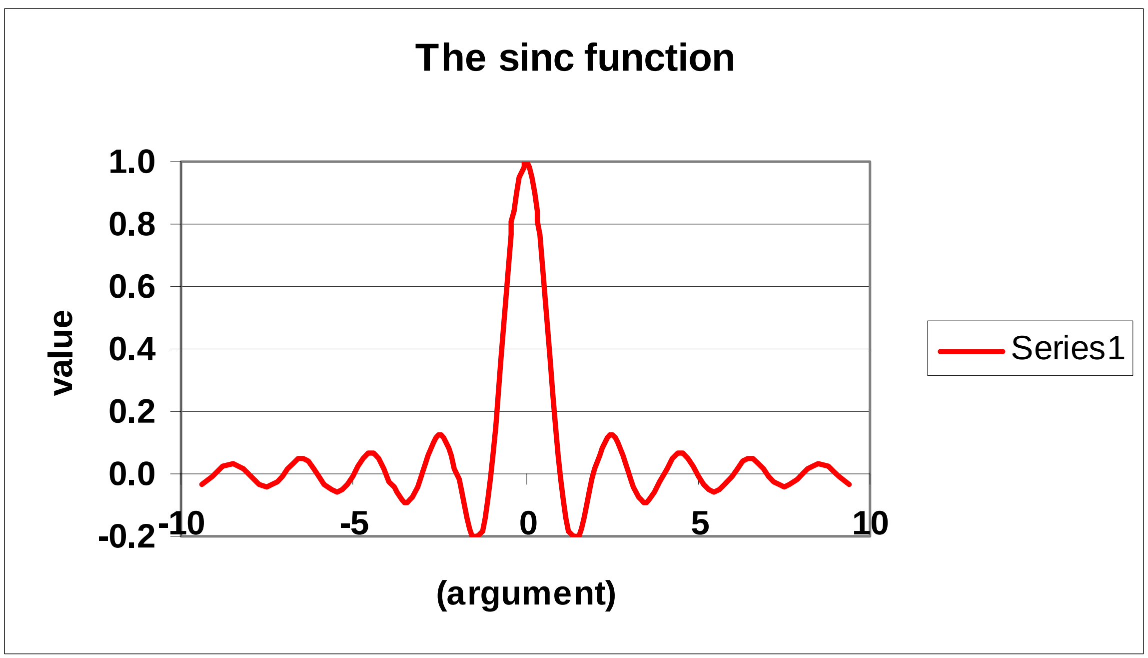
Evaluate at the limits:

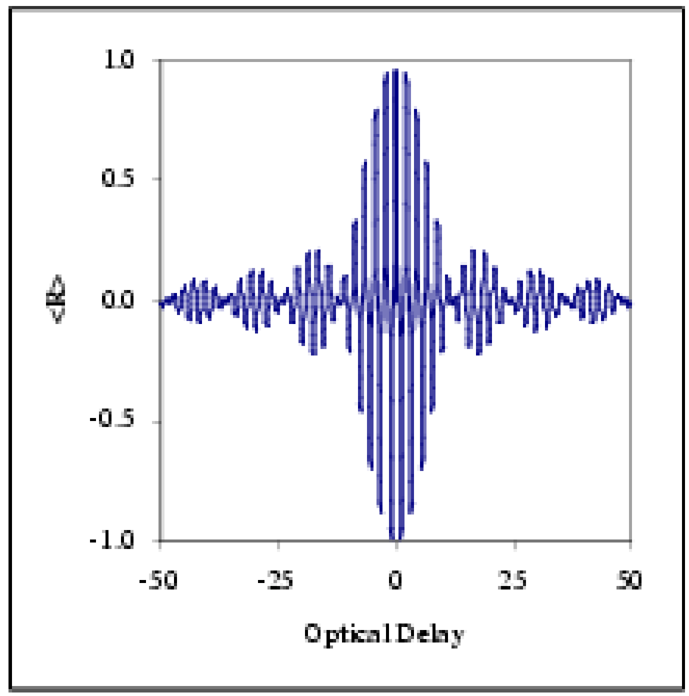
$$= \cos(2\pi\sigma_0\Delta)\left[\frac{\sin(\pi\delta_0\Delta)}{(\pi\delta_0\Delta)}\right]$$

Band-averaged response function

$$\langle R^P \rangle = \cos(2\pi\sigma_0\Delta)\frac{\sin(\pi\delta_0\Delta)}{(\pi\delta_0\Delta)}$$

$$= \cos(2\pi\sigma_0\Delta)\,\mathrm{sin}\,c(\pi\delta_0\Delta)$$

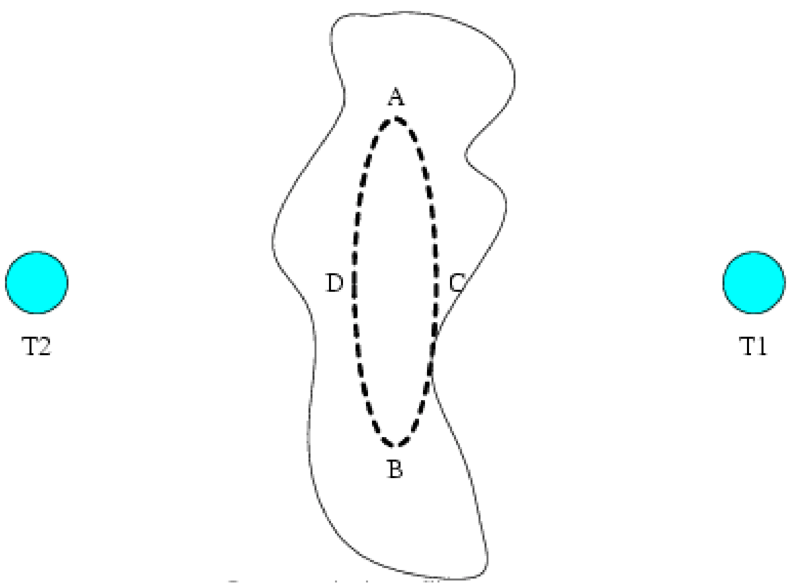




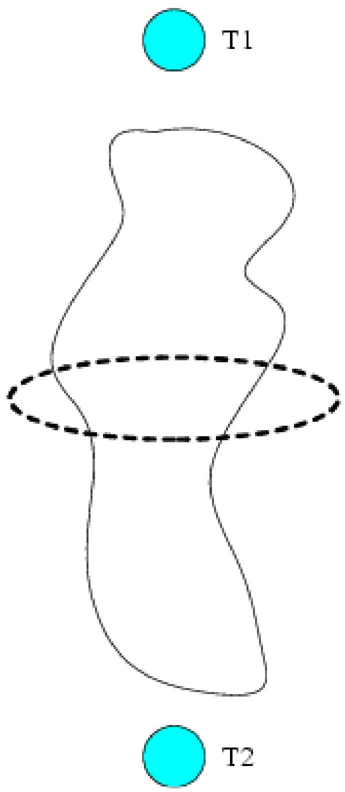
But, we want to measure a star's *size*!

We need to calculate the response to an extended source. Where to put the beam?

Beam undersamples object



Beam reoriented for better resolution



Extended sources

Done exactly the same as for a point source, but now we need to sum over all the point sources (that make up the extended source) within the beam.

$$I_\sigma(\Delta) = 2\left| \int_{-\infty}^{+\infty} \mathrm{I}_\sigma(\varepsilon) d\varepsilon + \int_{-\infty}^{+\infty} \mathrm{I}_\sigma(\varepsilon) \cos(2\pi\sigma)(\Delta+\varepsilon) d\varepsilon \right|$$

$\mathbf{F_{TOT}}$ + Intensity addition

Rearranging terms, the extended response becomes:

$$R_\sigma^E(\Delta) = \frac{I_\sigma(\Delta)}{2F_{TOT}} - 1 = \frac{1}{F_{TOT}} \int_{-\infty}^{+\infty} \mathrm{I}_\sigma(\varepsilon) \cos(2\pi\sigma)(\Delta+\varepsilon) d\varepsilon$$

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Just as in the case for a point-response, we have:

$$\left\langle R_\sigma^E(\Delta)\right\rangle = \frac{\int R_\sigma^E(\Delta)d\sigma}{\int (passband)d\sigma}$$

We use our rectangular filter function, and thus:

$$\left\langle R_\sigma^E(\Delta)\right\rangle = \frac{1}{F_{TOT}}\int_{-\infty}^{+\infty} \mathrm{I}_\sigma(\varepsilon)\left(\frac{1}{\delta_\sigma}\int_{\sigma_0-\delta_\sigma/2}^{\sigma_0+\delta_\sigma/2}\cos(2\pi\sigma)(\Delta+\varepsilon)\right)d\varepsilon$$

$$\left\langle R_\sigma^E(\Delta)\right\rangle = \frac{1}{F_{TOT}}\int_{-\infty}^{+\infty} \mathrm{I}_\sigma(\varepsilon)\left\langle R^P(\Delta+\varepsilon)\right\rangle d\varepsilon = \frac{1}{F_{TOT}}\mathrm{I} \otimes \left\langle R^P\right\rangle$$

The response to an extended source is the convolution of the point-response function with the intensity function of the source!

The visibility function, when transformed into the Fourier (or "UV") plane is defined as

$$V(u,v) = \iint_{-\infty}^{+\infty} I(\eta,\varepsilon) e^{-2\pi i(u\varepsilon+v\eta)} d\varepsilon d\eta$$

We take the Fourier transform

$$G(s) = \int_{-\infty}^{+\infty} g(x) e^{2\pi i x s} dx$$

which yields

$$|G|^2 = G(s)G(s)^* = I_1{}^2 + I_2{}^2 + 2I_1 I_2 \cos(2\pi\rho s)$$

and thus

the fringe visibility is the Fourier transform of the dual aperture function

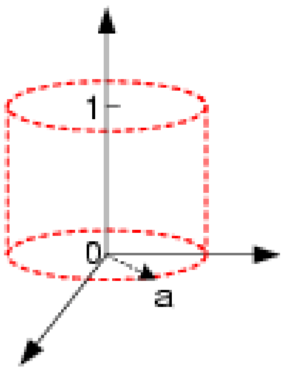
$$V^2 = \frac{G(s)G(s)^*}{G(0)G(0)^*}$$

Let's look at a "uniform disk"

Stars, *to first order*, are nothing more than a uniformly-bright circle on the sky as seen by the interferometer.

We introduce a "top-hat" function.

$$H(r/2a) = \begin{cases} 1 & r \le a \\ 0 & r > a \end{cases}$$

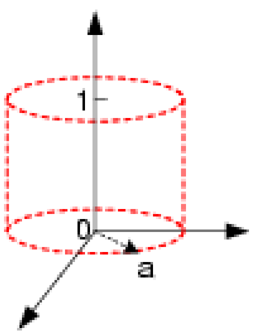


So, we insert a model geometry. In this case, a top-hat function (uniformly-bright disk) is used, with the substitution (z = 2πrs) yields

$$G(s) = \frac{1}{2\pi s^2}\int_0^{2\pi s a} zJ_0(z)dz$$

$$H(r/2a) = \begin{cases} 1 & r \leq a \\ 0 & r > a \end{cases}$$

Performing the integration and utilizing



$$V^2 = \frac{G(s)G(s)^*}{G(0)G(0)^*}$$

yields the visibility function for a uniform disk:

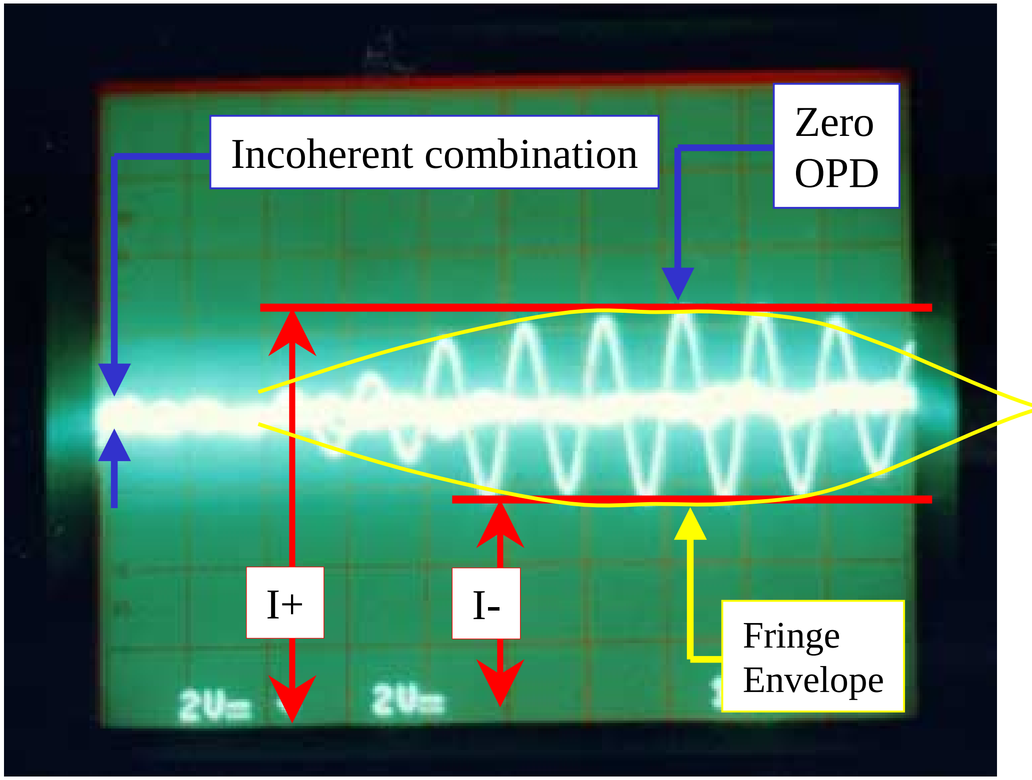
$$|V|^2 = \left(2\frac{J_1(\pi \mathrm{B}\,\theta / \lambda,)}{\pi \mathrm{B}\,\theta / \lambda}\right)^2$$

Fringe Visibility

- Constructive & destructive interference of light
- Fringe contrast or visibility:

$$V = \frac{I^+ - I^-}{I^+ + I^-}$$

- Calibration issues
  - Detector linearity
  - Zero point measurement
  - Noise characterization



Actual starlight fringes from IOTA - β And
Photo credit: R.R. Thompson

Visibility Function

- For a 'uniform disk', visibility matches:

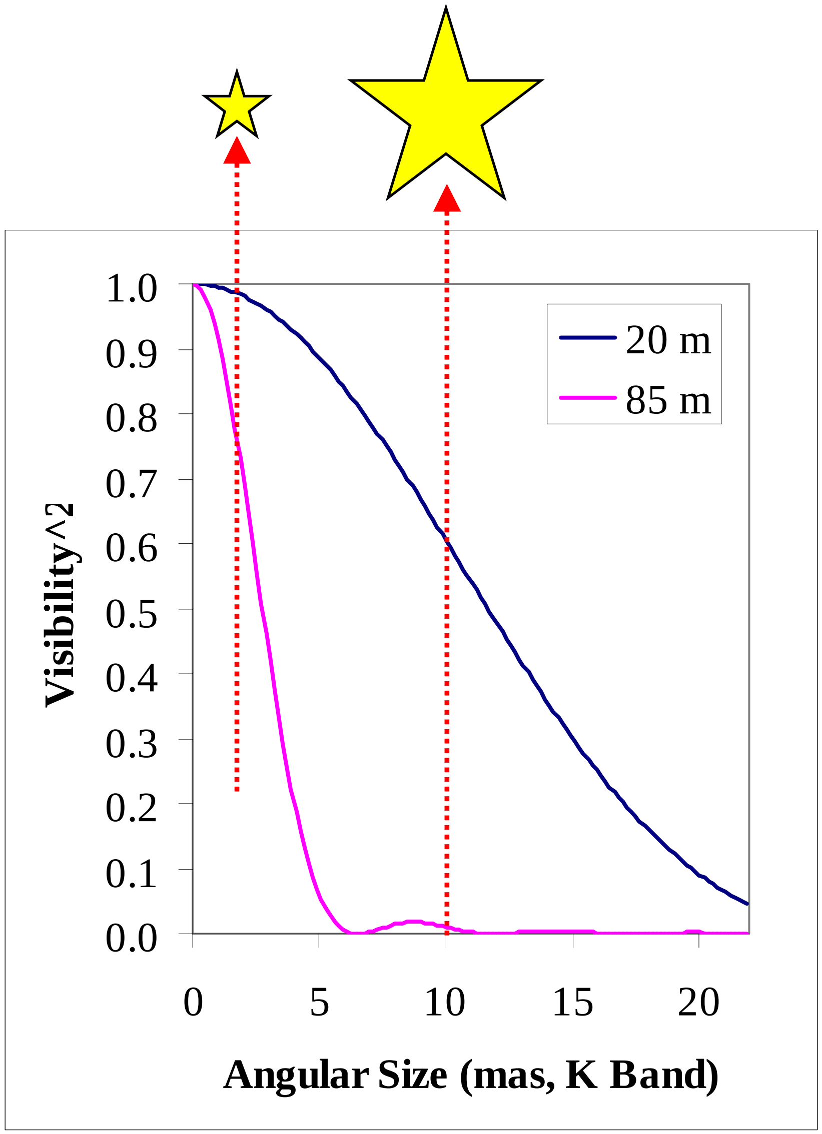
$$V = \frac{J_1(x)}{x} \quad \text{where} \quad x = \frac{\pi\theta B}{\lambda}$$

  B is the projected baseline

  θ is the stellar disk size

  λ is the instrumental wavelength

- Baseline, wavelength known
  - Can solve for θ
- Use V^2 instead of V
  - Unbiased estimator of visibility
  - See Colavita (1999)



Other Geometries

Geometry	Intensity function	Visibility Function	Parameters
Uniform disk	$H(r/2a)=\begin{cases}1\\0\end{cases}$	$\left(2\frac{J_1(\pi \mathrm{B}\theta/\lambda,)}{\pi \mathrm{B}\theta/\lambda}\right)^2$	H(r≤a)=1, H(r>a)=0
Gaussian Disk	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}$	$\exp\left(\frac{-4\pi^2}{8\ln 2}s^2\sigma^2\right)$	σ is the FWHM of the gaussian
Thin circular ring	$I\delta(r-a)$	$J_0{}^2(\pi ds)$	width << diameter, d
Two displaced point sources	$I_1\delta(x-\frac{1}{2}\rho)+I_2\delta(x+\frac{1}{2}\rho)$	$\left\lvert V_0{}^2+(1-V_0)^2+2V_0(1-V_0)\cos\gamma\right\rvert$	ρ = separation distance

$$V_0=\frac{I_1}{I_1+I_2} \qquad \gamma=2\pi s\rho$$

Binary stars

Let's say we wish to resolve a binary star orbit. We assume the two stars are uniform disks (UD), such that

$$|V|^2 = \left(2 \frac{J_1(\pi \mathrm{B}\, \theta / \lambda,)}{\pi \mathrm{B}\, \theta / \lambda} \right)^2$$

The expects squared visibility of a binary star is given by:

$$V_{nb}^2 = \frac{V_1^2 + V_2^2 r^2 + 2 V_1 V_2 r \cos\left(\frac{2\pi}{\lambda} \vec{B} \bullet \vec{s}\right)}{(1 + r^2)}$$

where V_1 and V_2 are the visibility moduli for the two components, r is the apparent brightness ratio, B is the projected baseline vector, and s is the primary-secondary angular separation vector on the plane of the sky.

Binary stars

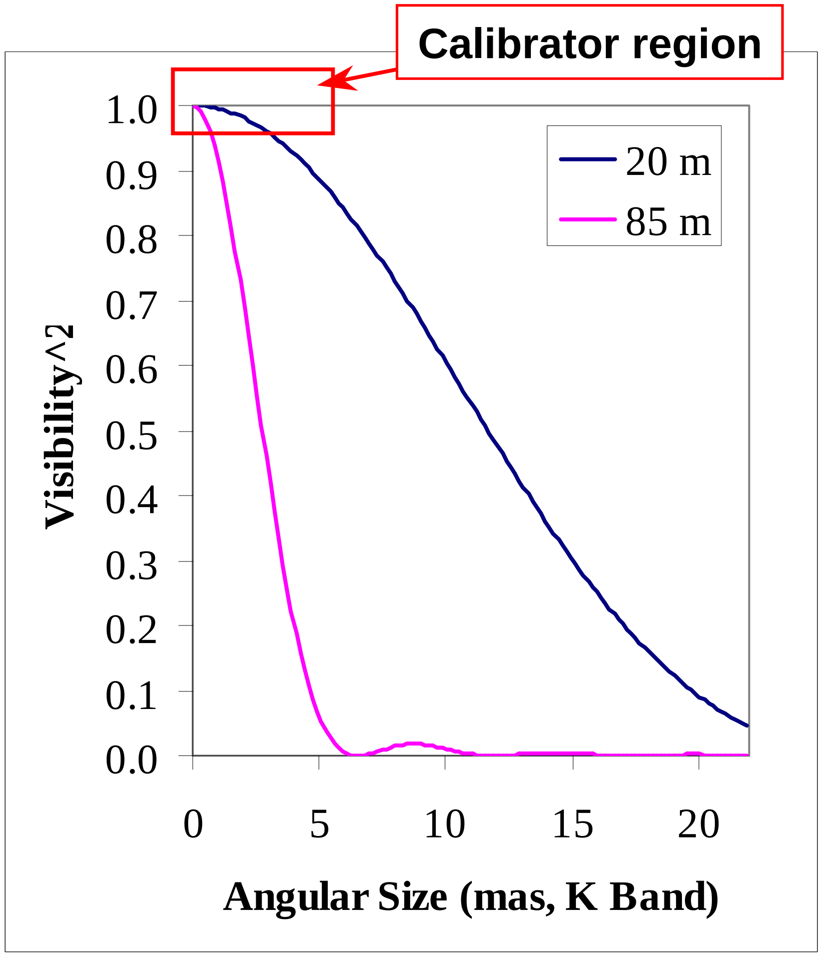
- Use the visibility data to test against a model orbit.
- Determine best-fit orbital parameters: separation, eccentricity, period, etc.
- Use Kepler's Laws and radial velocity data to determine star masses.
- From here, can investigate binary star evolution! (Do stars in binary systems evolve in the same way as a single star? BIG question in astrophysics today.)

Other geometries

- Stars with disks (YSOs, T Tauri stars, PPN, …)
- Multiple-star systems
- Departures from spherical symmetry: elongated stars
- Stellar photospheric phenomena: star spots, limb darkening/brightening, center-to-limb variation
- Multi-wavelength studies to determine chemistry/geometry

Visibility Function: Calibrators

- Atmospheric and instrumental effects reduce system V^2
- Observe 'unresolved' sources to establish system response
  - Use an estimate of size
  - Assume V^2 gains are equal
  - Gain factor is analogous to Strehl ratio
  - Flattening portion of visibility function $\rightarrow$ errors in calibrator size do not translate into errors in system V^2



Summary

- **The largest telescopes can resolve down to ~ 50 mas**
- **Thus, larger "apertures" are needed**
- **Interferometers with baselines ~ 100 m resolve to 1 mas at $\lambda = 1\ \mu m$**
- **The quantity measured by an interferometer is called the visibility**
- **The visibility is the Fourier transform of the dual aperture function**
- **The interpretation of the visibility is based on model-dependent assumptions (uniform disk, gaussian disk, 2 point sources, etc).**
- **Interferometers aren't perfect instruments: calibrator stars are needed to normalize target visibilities.**
- **Two-telescope interferometer data is entirely model-dependent (no reference for the phase information).**
- **Number of apertures (N) reduces model-dependence: fraction of information from observation goes as (N-2) / N**

The End....(?)